SOLUTION OF A CAUCHY PROBLEM FOR THE EQUATIONS OF HYDRAULIC SHOCK IN ELASTICALLY DEFORMABLE TUBES

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The problem of the occurrence of hydraulic shock in various engineering devices has long attracted the attention of investigators. For example, in [1] the hydraulic shock model of Zhukov was generalized with allowance for nonlinearity and friction in a quasistatic formulation. Here, hydraulic shock was considered to have been caused by a change in the flow regime at one end of a tubular conduit and was calculated by the method of successive approximations. The motion of a compressible viscous Newtonian fluid in an elastic shell was examined in [2] with allowance for its inertial properties. It was determined in [3] that the equation of state of water in the Tate form can be linearized at shock pressures lower than 100 MPa. In [4], the Zhukov model was used to examine hydraulic shock in cooling systems caused by the action of a distributed live load and inertial body forces on the system. The method of isolating a discontinuity of distributed live load (i.e., an examination of the conditions of mass and momentum conservation at a discontinuity) was used in [4] to construct an analytic solution in a linear approximation. In the present study, we show that this approach is acceptable only for subsonic regimes of motion of a live load. It was emphasized in [5] that Zhukov's model has made it possible to solve a number of complex engineering problems, including the propagation of waves in tubes of variable cross section and in coaxial tubes. However, it was also pointed out that this theory cannot explain wave dispersion, wavelike changes in pressure near fronts, etc. Other shortcomings of the Zhukov model include its unclosed nature, such as is manifest in problems involving resonance. The author of [6] examined asymptotic solutions for resonance regimes that are encountered in different engineering problems. In the present study, we use the method of characteristics within the framework of the Zhukov model of hydraulic shock to find an analytic solution to a Cauchy problem for equations describing a hydraulic shock caused by a distributed load traveling along the axis of an elastically deformable tube. Exact solutions are obtained for both the steady-state case and for linear resonance. The transition through resonance is described within the framework of an asymptotic approach. A comparative study is made of steady and unsteady solutions, as well as of methods of isolating and spreading out discontinuities of the distributed live load.

We will examine a long circular cylindrical tube containing a flow of an incompressible barotropic ideal fluid in drop form (water, for example). It should be noted that the mathematical model examined below also describes the case of two coaxial tubes when liquid circulates in the resulting gap and a live load travels over the inside surface of the inner tube [4] or the outside surface of the outer tube.

The behavior of the fluid in engineering devices conforming to the arrangement just given is described in a hydraulic approximation by the following system of equations [4]:

$$\frac{\partial \rho F}{\partial t} + \frac{\partial \rho u F}{\partial x} = 0; \qquad (1)$$

$$\frac{\partial \rho u F}{\partial t} + \frac{\partial}{\partial x}[\rho u^2 F + pF] = p\frac{\partial F}{\partial x}, \quad (p+B)/\rho^m = \text{const}, \qquad (2)$$
$$F = F_0 - A_1 P + A_2 p.$$

Here, ρ, p and u are the mean density, pressure, and velocity of the fluid in a fixed section of the channel with the stationary coordinate x; F is the cross-sectional area of the channel; P is the distributed live load; t is time; B and m are constants in the condition of barotropy of the fluid; F_0, A_1, and A_2 are constants in the equation for the area of the channel (formulas for these constants were presented in [4] for a system of coaxial tubes).

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Using the following relations to change over from a stationary coordinate system x to a system of coordinates X moving at the velocity V(t),

$$x = X + \xi(t), \quad \xi(t) = \int_0^t V(t)\,dt, \quad u = \omega + V(t) \tag{3}$$

and linearizing, we can represent hyperbolic equations (1), (2) in partial derivatives in the following characteristic form

$$\left[\frac{\partial}{\partial t} + (-V \pm c_0)\frac{\partial}{\partial X}\right]\left(\omega + V \pm \frac{p}{\rho_0 c_0} \mp c_0 \frac{A_1}{F_0} P\right) = -f(X, t), \tag{4}$$

$$f(X, t) = c_0^2 \frac{A_1}{F_0} \frac{\partial P(X, t)}{\partial X},$$

where ω is the mean velocity of the fluid in the moving coordinate system X; ρ_0 is the linearized value of the density of the fluid; c_0 is the linearized value of the speed of sound in the fluid in a channel with elastic walls (or the velocity of hydraulic shock in accordance with the Zhukov model).

With known values of p(X, 0) and ω(X, 0), the general solution of the Cauchy problem for system (4) has the form

$$\omega(X, t) = -V(t) + V(0) + \frac{1}{2}\left\{\omega(\tau, 0) + \omega(\eta, 0) + \frac{p(\tau, 0) + p(\eta, 0)}{\rho_0 c_0} + \right. \tag{5}$$

$$+ c_0 \frac{A_1}{F_0}[P(\eta, 0) - P(\tau, 0)] - \int_0^t [f(-\xi(z) - c_0 z + \eta, z) +$$

$$\left. + f(-\xi(z) + c_0 z + \tau, z)]\,dz\right\};$$

$$p(X, t) = \frac{\rho_0 c_0}{2}\left\{\omega(\tau, 0) - \omega(\eta, 0) + \frac{p(\tau, 0) + p(\eta, 0)}{\rho_0 c_0} + \right. \tag{6}$$

$$+ c_0 \frac{A_1}{F_0}[2P(X, t) - P(\tau, 0) - P(\eta, 0)] +$$

$$\left. + \int_0^t [f(-\xi(z) - c_0 z + \eta, z) - f(-\xi(z) + c_0 z + \tau, z)]\,dz\right\},$$

$$\tau = X + \xi(t) - c_0 t, \ \eta = X + \xi(t) + c_0 t.$$

Solutions (5) and (6) describe both free and forced oscillations of fluid in a channel with elastic walls. Forced vibrations develop under the influence of the live load P on a channel containing fluid. Free vibrations, due to the initial conditions, are not studied here. We will further assume that the load P can be represented in the form $P(X, t) = P_0(t)\cdot\varphi(X)$.

We will examine several modes of motion of P, making it possible to distinguish between the steady and unsteady solutions. The quadratures of solutions (5) and (6) can be calculated exactly for the first two regimes (steady-state regime and linear resonance). For the general case of motion of the load P, solutions infinite form can be obtained only asymptotically.

1. We obtain the steady-state regime with $P_0(t)$ = const, $V(t) = V_0$ = const and $V_0 \neq c_0$ ($M = V_0/c_0 \neq 1$) for t > 0. As an example, let us calculate the steady solution with

$$V(t) = V_0\theta(t), \ P(X, t) = R\theta(t)\varphi(X) = \theta(t)P_c(X), \tag{7}$$

where θ(t) = 1 at t ≥ 0; θ(t) = 0 at t < 0. Exact calculation of the integrals in (5)-(6) gives us the solution for forced vibrations in the form

$$\omega(X, t) = -V_0 + \frac{1}{2} c_0 \frac{A_1}{F_0}\left[\frac{2M^2}{M^2 - 1} P_c(X) - \frac{P_c(\tau)}{M - 1} - \frac{P_c(\eta)}{M + 1}\right],$$

$$p(X, t) = \frac{\rho_0 c_0^2}{2} \frac{A_1}{F_0}\left[\frac{2M^2}{M^2 - 1} P_c(X) - \frac{P_c(\tau)}{M - 1} + \frac{P_c(\eta)}{M + 1}\right], \quad M = V_0/c_0,$$

It follows from this that the steady solution depends on the steady load P_c and its velocity. The solution becomes singular at M = 1, so that it must be rejected from a physical view-

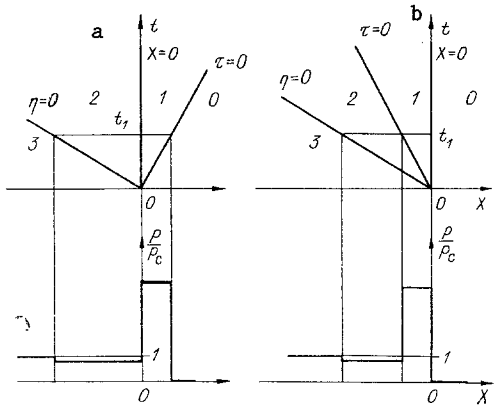

Fig. 1

point. In fact, it can be proven that there are no steady solutions for $M = 1$ (the linear resonance examined below takes place at $M = 1$). We will assign the function $\varphi(X)$ in the form of a spread-out step by means of the expression

$$\varphi(X) = \frac{1}{2}\left[1 - \operatorname{erf}\left(X\sqrt{\frac{n}{2}}\right)\right], \quad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-y^2)\,dy.$$

At $n \to \infty$ $\varphi(X) \to \theta(-X)$, i.e., the function $\varphi(X)$ becomes the step function, while the steady solution at this point is the regular generalized similarity solution for which Fig. 1 shows the wave patterns and distribution of pressure p. Here, the pressure is referred to $p_c = \rho_0 c_0{}^2(A_1/F_0)R$ for the subsonic (*a*, $M < 1$) and supersonic (b, $M > 1$) regimes of motion of the load P. The asymptote $n \to \infty$ signifies the transition to the method of isolation of a discontinuity in the load P [4]. This transition is proper from a mathematical viewpoint (leading to regular generalized functions), as well as from a physical viewpoint (the solutions are finite). It should be noted that the value of n does not affect the maximum of the solution in the given case. Thus, the method of spreading of the discontinuity in the load P (method of through computing), for which n is finite, gives the same result as the method of isolation of a discontinuity of P ($n \to \infty$) when used to calculate the maxima of hydrodynamic parameters. Given sufficiently large t, the fronts of waves propagating over the characteristic curves of families I and II travel quite far and in the neighborhood of the discontinuity of P_c (section $X = 0$) have the steady solutions examined in [7].

Assigning conditions similar to (7), we can also obtain other steady solutions.

2. A linear resonance occurs at $P_0(t) = \text{const}$, $V(t) = V_0 = \text{const}$ and $V_0 = c_0$ ($M = 1$) for $t > 0$. To calculate the solution in this case, as above we use conditions (7). Here, we assume that $V_0 = c_0$ in these conditions. Exactly calculating the integrals in (5) and (6), we find the following solution for forced vibrations during linear resonance

$$\omega(X, t) = c_0\left\{-1 - \frac{A_1}{2F_0}\left[\frac{\partial P_c}{\partial X}c_0 t + \frac{P_c(\eta) - P_c(X)}{2}\right]\right\},$$

$$p(X, t) = \frac{\rho_0 c_0^2}{2}\frac{A_1}{F_0}\left\{\frac{3}{2}P_c(X) + \frac{1}{2}P_c(\eta) - \frac{\partial P_c}{\partial X}c_0 t\right\},$$

which, in contrast to the solution obtained above, contains a nontrivial unsteady term, where $\partial P_c/\partial X \neq 0$. At these points, the parameters increase linearly over time, and at $t \to \infty$, $p \to \infty$, $\omega \to \infty$ (it can be seen from the numerical calculations for the nonlinear model that the hydrodynamic parameters turn out to be finite in nonlinear resonance). Calculating the derivative $\partial P_c/\partial X = -R\sqrt{\frac{n}{2\pi}}\exp(-n/2 \times X^2)$, we see that the unsteady part of the resulting solution is proportional to the square root of the number of the δ-like sequence n, i.e., it depends on how the load P is spread out. Since the unidimensional model does not given any information on the spreading, it is essentially unclosed, and its closure requires either experimental data or calculations for three dimensions. It is clear that n is finite for physically realistic motions. Thus, any method of through computing is actually the method of spreading out the discontinuity of the load P (for the isolation method, $n = \infty$). The asymptotic solution $n \to \infty$ is correct from a mathematical viewpoint, since it leads to

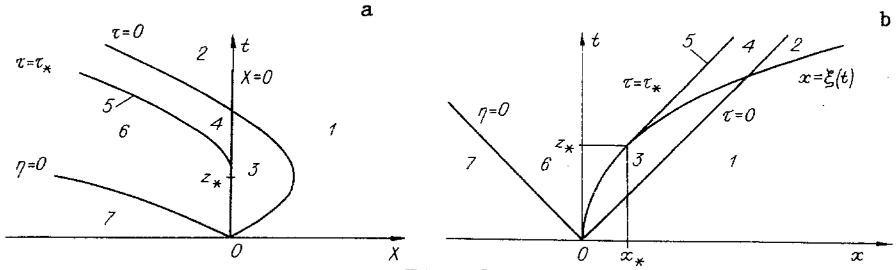

Fig. 2

singular generalized solutions. However, it is incorrect from a physical viewpoint, since it leads to infinite hydraulic shock pressures. The latter is inconsistent with both the physical meaning of the problem and with experimental results. This means that a linear resonance cannot be calculated by the method of discontinuity of the load P. To calculate a linear resonance by the method of spreading of the load P, it is necessary to know n. Having taken initial relations different from (7), we can also obtain other solutions for a linear resonance.

3. Let us examine the general case of motion of the load P. We impose the natural limitation W > 0 on the law of motion $\xi(t)$, i.e., we assume that the motion of P is accelerated. Given this restriction on $\xi(t)$, the characteristic curves of family I (τ = const) intersect X = 0 twice in the general case, while the curves of family II (η = const) intersect X = 0 once. We will calculate the asymptote of the integrals in solution (5), (6) at $n \to \infty$, when the front of the load P becomes discontinuous. In this case, the stationary points obtained with calculation of the integrals by the Laplace method are the points of intersection of the characteristic curves of families I and II with the law of motion (X = 0) of the discontinuity of the load P. The resulting solution has a generalized (piecewise-continuous) structure. In the relations presented below for the corresponding regions 1-7 (Fig. 2a), the equal sign is replaced by the asymptote sign, while the exponentially small terms that approach zero as $n \to \infty$ are discarded:

1) $\tau > 0 \cap X > 0$:

$$\omega(X, t) \sim -V(t),\ p(X, t) \sim 0;$$

2) $\tau > 0 \cap X < 0$:

$$\omega(X, t) \sim -V(t) + \frac{1}{2} c_0 \frac{A_1}{F_0} \left[\frac{P_0(z_1)}{M(z_1) - 1} + \frac{P_0(z_2)}{M(z_2) + 1} \right],$$

$$p(X, t) \sim \rho_0 c_0^2 \frac{A_1}{F_0} \left\{ P_0(t) + \frac{1}{2} \left[\frac{P_0(z_1)}{M(z_1) - 1} - \frac{P_0(z_2)}{M(z_2) + 1} \right] \right\};$$

3) $\tau_* > \tau > 0 \cap X > 0$:

$$\omega(X, t) \sim -V(t) + \frac{1}{2} c_0 \frac{A_1}{F_0} \frac{P_0(z_3)}{1 - M(z_3)}, \quad p(X, t) \sim \frac{1}{2} \rho_0 c_0^2 \frac{A_1}{F_0} \frac{P_0(z_3)}{1 - M(z_3)};$$

4) $\tau_* > \tau > 0 \cap X < 0 \cap t > z_*$:

$$\omega(X, t) \sim -V(t) + \frac{1}{2} c_0 \frac{A_1}{F_0} \left[\frac{P_0(z_3)}{1 - M(z_3)} + \frac{P_0(z_1)}{M(z_1) - 1} + \frac{P_0(z_2)}{M(z_2) + 1} \right],$$

$$p(X, t) \sim \rho_0 c_0^2 \frac{A_1}{F_0} \left\{ P_0(t) + \frac{1}{2} \left[\frac{P_0(z_3)}{1 - M(z_3)} + \frac{P_0(z_1)}{M(z_1) - 1} - \frac{P_0(z_2)}{M(z_2) + 1} \right] \right\};$$

5) $\tau = \tau_* \cap t > z_*$:

$$\omega(X, t) \sim -V(t) + \frac{1}{2} c_0 \frac{A_1}{F_0} \left[P_0(z_*) \sqrt[4]{n} \sqrt{\frac{c_0^2}{W(z_*)}} 1.2163 + \frac{P_0(z_2)}{M(z_2) + 1} \right],$$

$$p(X, t) \sim \rho_0 c_0^2 \frac{A_1}{F_0} \left\{ P_0(t) + \frac{1}{2} \left[P_0(z_*) \sqrt[4]{n} \sqrt{\frac{c_0^2}{W(z_*)}} 1.2163 - \frac{P_0(z_2)}{M(z_2) + 1} \right] \right\};$$

6) $(0 < \eta < \eta_* \cap X < 0) \cup (\eta > \eta_* \cap \tau < \tau_*)$:

$$\omega(X, t) \sim -V(t) + \frac{1}{2} c_0 \frac{A_1}{F_0} \frac{P_0(z_2)}{M(z_2) + 1},$$

$$p(X, t) \sim \rho_0 c_0^2 \frac{A_1}{F_0} \left[P_0(t) - \frac{1}{2} \frac{P_0(z_2)}{M(z_2) + 1} \right];$$

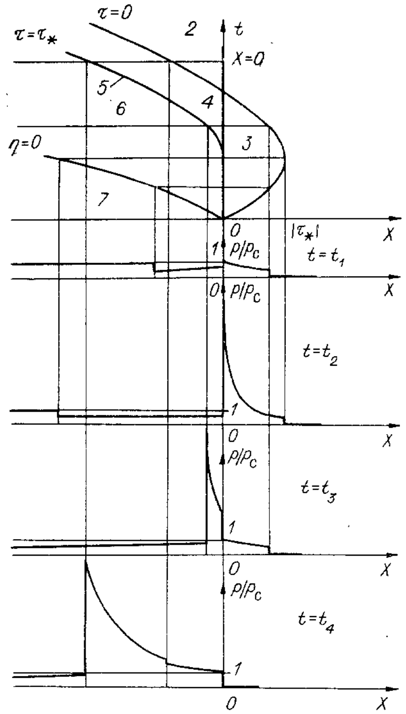

Fig. 3

7) $\eta < 0$:

$$\omega(X, t) \sim -V(t), \quad p(X, t) \sim \rho_0 c_0^2 \frac{A_1}{F_0} P_0(t).$$

Here, $M(z) = V(z)/c_0$;

$$z_1 = z_1(\tau),\ z_2 = z_2(\eta),\ z_3 = z_3(\tau); \tag{8}$$

where z_1 and z_3 are the larger and smaller values of the time of intersection of the curves of family I with X = 0; z_2 is the time of intersection of the curves of family II with X = 0.

At $n \to \infty$, the resulting generalized solution becomes a singular generalized solution (see Part 2). The values of the hydrodynamic parameters are singular on the characteristic curve $\tau = \tau_*$ at $t > z_*$. Thus, the method of isolation of the discontinuity of the load P cannot be used to calculate the transition through M = 1 and the subsequent motion. To do this by the method of spreading of the discontinuity of P, it is necessary to determine the value of n. This can be done by comparing the calculations from the unidimensional model with the results of an experiment or calculations performed for three dimensions, i.e., it can be done by solving a problem of parametric identification for the given mechanical system.

For a system of finite length, it is necessary to allow for the left and right boundary conditions. In this case, the solution in Part 3 needs to be written by means of Eqs. (3) in the moving coordinate system (Fig. 2b) and to be modified so that it contains arbitrary functions τ and η for satisfaction of the right and left boundary conditions.

It follows from the solution in Part 3 with finite n that the hydrodynamic parameters on the characteristic curve $\tau = \tau_*$ at $t > z_*$ depend on the acceleration at the point $(0, z_*)$ of the plane (X, t) (see Fig. 2a). For other regions, the solution depends on the load P and its relative velocity $M = V/c_0$, as for steady or regular generalized similarity solutions. Thus, in [8] these solutions were referred to as being quasisimilar. In essence, this term signifies solutions obtained by the method of isolation of a discontinuity of the load P ($n = \infty$). The quasisimilar solutions that exist at M < 1 are limiting (yield the maximum values of shock pressure) in relation to the actual motions of the load P. Comparing the equations in Parts 1 and 3 for region 3, we see that the values of p and ω on the corresponding characteristic curves τ = const in region 3 will be the same as for regular generalized similarity solutions with the values of P and V at the moment of the first intersection of the characteristic curve of family I and the law of motion of the load P (X = 0).

Comparing the solutions in Parts 2 and 3, we see that with any finite n the maximum pressures (in region 5) will be finite at any moment of time for accelerated motion of the load $P(W(z_*) \neq 0)$ and that $P_0(z_*)$ will also be finite. Meanwhile, this will be true for both linear resonance and for finite n when $t \to \infty$, $p \to \infty$. The dependence on n in the case of linear resonance ($\sqrt{n}$ enters into the formulas in Part 2) is greater than with an accelerated transition through M = 1 ($\sqrt[4]{n}$ goes into the formulas for region 5 in Part 3).

In the general case, Eqs. (8) can be obtained only numerically (or graphically which is considerably simpler) on the basis of the solution of the nonlinear equations $\xi(z) = c_0 z + \tau$, $\xi(z) = -c_0 z + \eta$. The exact solution of these equations can be found, for example, in the case of equal accelerated motion of a discontinuity of the load P with the law of motion $\xi = Wt^2/2$:

$$z_1 = \frac{c_0}{W} + \sqrt{\left(\frac{c_0}{W}\right)^2 + \frac{2\tau}{W}},$$

$$z_2 = -\frac{c_0}{W} + \sqrt{\left(\frac{c_0}{W}\right)^2 + \frac{2\eta}{W}},$$

$$z_3 = \frac{c_0}{W} - \sqrt{\left(\frac{c_0}{W}\right)^2 + \frac{2\tau}{W}}.$$

Assuming $P(X, t) = R\theta(t)\varphi(X)$, we obtain the solution in Fig. 3, which can readily be analyzed by the methods of mathematical analysis. At $n \to \infty$, $\varphi(X) \to \theta(-X)$ will also coincide with the solution in [8], obtained by the method of isolation of a discontinuity of the load P at M < 1.

Figure 3 shows the distribution of fluid pressure in the moving coordinate system for $t_1 = c_0/2W$ (M = 1/2), $t_2 = c_0/W$ (M = 1), $t_3 = 3/2 \times c_0/W$ (M = 3/2), $t_4 = 5/2 \times c_0/W$ (M = 5/2) and wave configuration associated with the asymptotic solution for equal accelerated motion of a steady load P; 1-7 are regions with a continuous distribution of pressure and fluid velocity, $\tau_* = -c_0^2/2W$.

The above analytical results can be used to refine and test numerical algorithms. They also help us understand certain features of numerical solutions of unidimensional hydraulic-shock problems that arise in the motion of a discontinuity of a load P.

In conclusion, we noted that the problem of nonclosure arises only in a unidimensional description of hydraulic shock and that the unidimensional model gives only one value of critical velocity at which resonance takes place.

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